

UNIT CIRCLE ELLIPTIC BETA INTEGRALS

J.F. VAN DIEJEN AND V.P. SPIRIDONOV

ABSTRACT. We present some elliptic beta integrals with a base parameter on the unit circle, together with their basic degenerations.

1. INTRODUCTION

The theory of generalized gamma functions has been set up by Barnes [Ba]. A slightly different approach was advocated by Jackson, who considered the basic gamma function depending on one base parameter q and the elliptic gamma function depending (symmetrically) on two bases p and q [J]. For a long time, only the first of these generalized gamma functions was appreciated in the literature [AAR]. Recently, however, the elliptic gamma function also got appropriate attention after the work of Ruijsenaars [Ru1], who introduced it in the context of integrable systems and investigated some of its properties. A further study of the function in question was conducted by Felder and Varchenko [FV]. A modified elliptic gamma function, which admits analytic continuation in one of the base parameters, e.g. q , onto the unit circle $|q| = 1$, has been introduced recently by one of us in [S2].

In this paper we study beta type integrals on the unit circle $|q| = 1$ built of modified elliptic gamma functions, as well as their basic degenerations. The first exact beta type integration formula involving the conventional elliptic gamma function was discovered in [S1]. Various multidimensional generalizations of this elliptic beta integral associated with the C_N and A_N root systems have been investigated in [DS1, DS2, R, S2]. A general theory of theta hypergeometric integrals on tori and the Jacobi theta function generalizations of the Meijer function was developed in [S2]. The beta integrals considered below should be thought of as the $|q| = 1$ counterparts of the integrals in [S1] and [DS1]. Recently, the Bailey's technique of deriving identities for series of hypergeometric type [AAR] has been generalized to the level of integrals [S3]. It can also be extended to the $|q| = 1$ integrals under discussion.

2. PRELIMINARIES:

THE JACOBI THETA FUNCTION AND THE ELLIPTIC GAMMA FUNCTION

The main underlying structural object of this paper is a Jacobi type theta function defined as

$$\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty, \quad (a; p)_\infty = \prod_{n=0}^{\infty} (1 - ap^n), \quad (1)$$

with $z, p \in \mathbb{C}$, $|p| < 1$. It satisfies the transformation properties

$$\theta(pz; p) = \theta(z^{-1}; p) = -z^{-1}\theta(z; p). \quad (2)$$

Date: August 2003; *Ramanujan J.*, to appear.

Evidently, $\theta(z; p) = 0$ for $z = p^m$, $m \in \mathbb{Z}$, and $\theta(z; 0) = 1 - z$. If we denote $p = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$, then the standard Jacobi θ_1 -function [WW] is related to $\theta(z; p)$ as

$$\begin{aligned}\theta_1(u|\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n p^{(2n+1)^2/8} e^{\pi i (2n+1)u} \\ &= p^{1/8} i e^{-\pi i u} (p; p)_{\infty} \theta(e^{2\pi i u}; p), \quad u \in \mathbb{C}.\end{aligned}\quad (3)$$

The complex function $\theta_1(u|\tau)$ is entire, odd, and doubly quasiperiodic in u

$$\begin{aligned}\theta_1(u+1|\tau) &= -\theta_1(u|\tau), \\ \theta_1(u+\tau|\tau) &= -e^{-\pi i \tau - 2\pi i u} \theta_1(u|\tau).\end{aligned}\quad (4)$$

The modular $PSL(2, \mathbb{Z})$ -group,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad u \rightarrow \frac{u}{c\tau + d}, \quad (5)$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, is generated by the two transformations $\tau \rightarrow \tau + 1$, $u \rightarrow u$ and $\tau \rightarrow -\tau^{-1}$, $u \rightarrow u\tau^{-1}$. Its action on the Jacobi theta function is determined by

$$\theta_1(u|\tau+1) = e^{\pi i/4} \theta_1(u|\tau), \quad (6a)$$

$$\theta_1\left(\frac{u}{\tau} \middle| -\frac{1}{\tau}\right) = -i(-i\tau)^{1/2} e^{\pi i u^2/\tau} \theta_1(u|\tau). \quad (6b)$$

(Throughout this paper the sign of the square root is fixed in accordance with the principal branch with the cut chosen on the negative real axis.) From the second of these relations, combined with the modular transformation law for the Dedekind η -function

$$e^{-\frac{\pi i}{12\tau}} \left(e^{-2\pi i/\tau}; e^{-2\pi i/\tau} \right)_{\infty} = (-i\tau)^{1/2} e^{\frac{\pi i \tau}{12}} \left(e^{2\pi i \tau}; e^{2\pi i \tau} \right)_{\infty}, \quad (7)$$

one readily deduces a corresponding modular transformation formula for the $\theta(z; p)$ function

$$\theta(e^{2\pi i \frac{u}{\tau}}; e^{-2\pi i \frac{1}{\tau}}) = -i e^{\pi i (\frac{u^2}{\tau} + \frac{\bar{u}}{6} + \frac{1}{6\tau} + \frac{u}{\tau} - u)} \theta(e^{2\pi i u}; e^{2\pi i \tau}). \quad (8)$$

The elliptic gamma function $\Gamma(z; q, p)$, $|q|, |p| < 1$, is related to the above theta function through the difference equations

$$\Gamma(qz; q, p) = \theta(z; p) \Gamma(z; q, p), \quad \Gamma(pz; q, p) = \theta(z; q) \Gamma(z; q, p). \quad (9)$$

It is given by the explicit product representation [Ru1]

$$\Gamma(z; q, p) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} q^{j+1} p^{k+1}}{1 - z q^j p^k}. \quad (10)$$

From this representation the following reflection relation is immediate

$$\Gamma(z; q, p) \Gamma(z^{-1}; q, p) = \frac{1}{\theta(z; p) \theta(z^{-1}; q)}. \quad (11)$$

3. THE MODIFIED ELLIPTIC GAMMA FUNCTION

The modified elliptic gamma function introduced in [S2] is constructed as a product of two elliptic gamma functions of the form in (10), corresponding to two different choices of bases. It is convenient to pass to an additive formulation by introducing three pairwise incommensurate quasiperiods $\omega_1, \omega_2, \omega_3$ and write

$$\begin{aligned} q &= e^{2\pi i \frac{\omega_1}{\omega_2}}, & p &= e^{2\pi i \frac{\omega_3}{\omega_2}}, & r &= e^{2\pi i \frac{\omega_3}{\omega_1}}, \\ \tilde{q} &= e^{-2\pi i \frac{\omega_2}{\omega_1}}, & \tilde{p} &= e^{-2\pi i \frac{\omega_2}{\omega_3}}, & \tilde{r} &= e^{-2\pi i \frac{\omega_1}{\omega_3}}, \end{aligned} \quad (12)$$

(i.e., $\tau = \omega_3/\omega_2$). The tilded bases \tilde{q}, \tilde{p} , and \tilde{r} are the respective modular transformations of q, p , and r . For $\text{Im}(\omega_1/\omega_2), \text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$ (so $|q|, |p|, |r| < 1$), the modified elliptic gamma function is now defined as [S2]

$$G(u; \omega) = \prod_{j,k=0}^{\infty} \frac{(1 - e^{-2\pi i \frac{u}{\omega_2}} q^{j+1} p^{k+1})(1 - e^{2\pi i \frac{u}{\omega_1}} \tilde{q}^{j+1} r^k)}{(1 - e^{2\pi i \frac{u}{\omega_2}} q^j p^k)(1 - e^{-2\pi i \frac{u}{\omega_1}} \tilde{q}^j r^{k+1})}. \quad (13)$$

It satisfies three difference equations

$$G(u + \omega_1; \omega) = \theta(e^{2\pi i \frac{u}{\omega_2}}; p) G(u; \omega), \quad (14a)$$

$$G(u + \omega_2; \omega) = \theta(e^{2\pi i \frac{u}{\omega_1}}; r) G(u; \omega), \quad (14b)$$

$$G(u + \omega_3; \omega) = \frac{\theta(e^{2\pi i \frac{u}{\omega_2}}; q)}{\theta(e^{2\pi i \frac{u}{\omega_1}} \tilde{q}; \tilde{q})} G(u; \omega), \quad (14c)$$

the latter of which can be rewritten with the aid of modular transformation (8) as

$$G(u + \omega_3; \omega) = e^{-\pi i B_{2,2}(u; \omega)} G(u; \omega), \quad (14d)$$

where

$$B_{2,2}(u; \omega) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

Such a system of three difference equations determines the meromorphic function $G(u; \omega)$ up to a multiplicative constant (which is the only meromorphic function with three pairwise incommensurate periods $\omega_{1,2,3}$). Similar to the $\Gamma(z; q, p)$ function, the $G(u; \omega)$ function satisfies a simple reflection equation given by

$$G(u; \omega) G(-u; \omega) = \frac{e^{\pi i B_{2,2}(u; \omega)}}{\theta(e^{-2\pi i \frac{u}{\omega_1}}; r) \theta(e^{-2\pi i \frac{u}{\omega_2}}; p)}. \quad (15)$$

If we fix the quasiperiods ω_1, ω_2 such that $\text{Im}(\omega_1/\omega_2) > 0$, and take ω_3 to infinity in such a way that $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) \rightarrow +\infty$ (so $p, r \rightarrow 0$), then we obtain

$$\lim_{p, r \rightarrow 0} \frac{1}{G(u; \omega)} = S(u; \omega_1, \omega_2) = \frac{(e^{2\pi i u/\omega_2}; q)_{\infty}}{(e^{2\pi i u/\omega_1} \tilde{q}; \tilde{q})_{\infty}}. \quad (16)$$

In the modern time, the function $S(u; \omega)$ was introduced by Shintani [Sh]. It is related to the Barnes double gamma function and is called the double sine function [Ku]. Its various properties and applications are discussed, e.g., in [F, JM, KLS, M, NU, Ru1, Ru2]. Some q -beta integrals expressed in terms of $S(u; \omega)$ were considered in [FKV, PT, St, T].

One of the main properties of the double sine function consists of the fact that it can be extended continuously in the quasiperiods $\omega_{1,2}$ from the upper half plane $\text{Im}(\omega_1/\omega_2) > 0$ (so $|q| < 1$) to the positive real axis $\omega_1/\omega_2 > 0$ (so $|q| = 1$), the resulting function still being meromorphic in u . A similar situation holds for

the function $G(u; \omega)$, as can be verified by expressing it in terms of the Barnes' multiple gamma function of the third order [S2]. The following theorem provides a convenient representation of $G(u; \omega)$ detailing explicitly its structure when q lies on the unit circle.

Theorem 1. *Let $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$ and $\text{Im}(\omega_1/\omega_2) \geq 0$. The analytic continuation of $G(u; \omega)$ (14d) from the domain $\text{Im}(\omega_1/\omega_2) > 0$ to the boundary $\omega_1/\omega_2 > 0$ is given by the meromorphic function*

$$G(u; \omega) = e^{-\pi i P(u)} \Gamma(e^{-2\pi i \frac{u}{\omega_3}}; \tilde{r}, \tilde{p}), \quad (17a)$$

where $P(u)$ is the following polynomial of the third degree

$$P(u) = \frac{1}{3\omega_1\omega_2\omega_3} \left(u - \frac{1}{2} \sum_{n=1}^3 \omega_n \right) \left(u^2 - u \sum_{n=1}^3 \omega_n + \frac{\omega_1\omega_2\omega_3}{2} \sum_{n=1}^3 \frac{1}{\omega_n} \right). \quad (17b)$$

Proof. Let us first assume temporarily that $\text{Im}(\omega_1/\omega_2) > 0$. We denote the right-hand side of (17a) by $f(u)$. It is easy to see that

$$\frac{f(u + \omega_1)}{f(u)} = e^{\pi i (P(u) - P(u + \omega_1))} \theta(e^{-2\pi i \frac{u}{\omega_3}}; \tilde{p}) = \theta(e^{2\pi i \frac{u}{\omega_2}}; p), \quad (18)$$

as a consequence of the modular transformation for theta functions in (8). The function $f(u)$ thus satisfies equation (14a). By symmetry, it satisfies (14b) as well. Analogously, we have that $f(u + \omega_3)/f(u) = e^{\pi i (P(u) - P(u + \omega_3))}$, which is seen to coincide with (14d). The independence of the quasiperiods $\omega_{1,2,3}$ over \mathbb{Q} now implies that $G(u; \omega)/f(u)$ must be constant. Its value is equal to one, because for $u = (\omega_1 + \omega_2 + \omega_3)/2$ we have that $G(u; \omega) = f(u) = 1$. The theorem then follows upon analytic continuation of the right-hand side of (17a) to the positive real axis $\omega_1/\omega_2 > 0$. \square

Remark 1. In [FV], modular transformation properties of the elliptic gamma function were investigated. For $|q|, |p|, |r| < 1$ the equality in (17a) follows from one of these modular transformations. Function $\Gamma(z; q, p)$ has a pointwise limit for some particular values of $\omega_1/\omega_2 \in X \subset \mathbb{R}_+$ [FV], but it does not assume validity of (17a) in this regime. Our result consists thus of the observation that, after an appropriate rewriting, this modular transformation provides a representation for the elliptic gamma function that is well defined for all $\omega_1/\omega_2 > 0$ [S2].

Below we shall need functional equations satisfied by $S(u; \omega)$

$$\frac{S(u + \omega_1; \omega)}{S(u; \omega)} = \frac{1}{1 - e^{2\pi i \frac{u}{\omega_2}}}, \quad \frac{S(u + \omega_2; \omega)}{S(u; \omega)} = \frac{1}{1 - e^{2\pi i \frac{u}{\omega_1}}} \quad (19)$$

and its asymptotics for u going to infinity. Let us fix the incommensurate quasiperiods ω_1 and ω_2 such that $\text{Im}(\omega_1/\omega_2) > 0$ (so $|q| < 1$). Then it follows from the infinite product representation (16) and its modular inverse

$$S(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u; \omega)} \frac{(e^{-2\pi i u/\omega_1}; \tilde{q})_\infty}{(e^{-2\pi i u/\omega_2} q; q)_\infty} \quad (20)$$

that

$$\lim_{\text{Im}(\frac{u}{\omega_1}), \text{Im}(\frac{u}{\omega_2}) \rightarrow +\infty} S(u; \omega) = 1, \quad (21a)$$

$$\lim_{\text{Im}(\frac{u}{\omega_1}), \text{Im}(\frac{u}{\omega_2}) \rightarrow -\infty} e^{\pi i B_{2,2}(u; \omega)} S(u; \omega) = 1. \quad (21b)$$

It turns out that the same asymptotics also holds for the boundary domain $\omega_1/\omega_2 > 0$ (so $|q| = 1$), as can be verified by means of an integral representation for $S(u; \omega)$ [KLS, Ru1].

Remark 2. If we take the limit $p, r \rightarrow 0$ in (17a), then the transition from $G(u; \omega)$ to the double sine function simplifies to

$$\begin{aligned} \lim_{\substack{\text{Im}(\frac{\omega_3}{\omega_1}), \text{Im}(\frac{\omega_3}{\omega_2}) \rightarrow +\infty}} \left(e^{-\pi i \omega_3 \frac{2u - \omega_1 - \omega_2}{12\omega_1\omega_2}} \Gamma(e^{-2\pi i \frac{u}{\omega_3}}; e^{-2\pi i \frac{\omega_3}{\omega_1}}, e^{-2\pi i \frac{\omega_3}{\omega_2}}) \right) \\ = e^{-\pi i \frac{3(2u - \omega_1 - \omega_2)^2 - \omega_1^2 - \omega_2^2}{24\omega_1\omega_2}} S^{-1}(u; \omega_1, \omega_2). \end{aligned} \quad (22)$$

Such a limiting relation was first derived in a different way and in a stronger sense by Ruijsenaars [Ru1].

4. THE ELLIPTIC BETA INTEGRAL ON THE UNIT CIRCLE

We now turn to the elliptic beta integrals. Let $|q|, |p| < 1$ and let t_n , $n = 0, \dots, 4$, be five complex parameters satisfying the inequalities $|t_n| < 1$ and $|pq| < |A|$, where $A \equiv \prod_{n=0}^4 t_n$. The elliptic beta integral of [S1] states that

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\prod_{n=0}^4 \Gamma(t_n z^{\pm}; q, p)}{\Gamma(z^{\pm 2}, Az^{\pm}; q, p)} \frac{dz}{z} = \frac{2 \prod_{0 \leq n < m \leq 4} \Gamma(t_n t_m; q, p)}{(q; q)_{\infty} (p; p)_{\infty} \prod_{n=0}^4 \Gamma(At_n^{-1}; q, p)}, \quad (23)$$

where \mathbb{T} denotes the positively oriented unit circle. Here we have employed the shorthand notations

$$\begin{aligned} \Gamma(z_1, \dots, z_m; q, p) &\equiv \prod_{l=1}^m \Gamma(z_l; q, p), \\ \Gamma(tz^{\pm}; q, p) &\equiv \Gamma(tz, tz^{-1}; q, p), \quad \Gamma(z^{\pm 2}; q, p) \equiv \Gamma(z^2, z^{-2}; q, p). \end{aligned}$$

For $p = 0$, the integration formula (23) amounts to an integral explicitly constructed by Rahman in [Rah] through a specialization of the Nassrallah-Rahman integral representation for a very-well-poised basic hypergeometric ${}_8\phi_7$ series.

The following theorem provides a modified elliptic beta integral that—unlike (23)—is well defined for $|q| = 1$.

Theorem 2. *Let $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$, and let g_n , $n = 0, \dots, 4$, be five complex parameters subject to the constraints*

$$\text{Im}(g_n/\omega_3) < 0, \quad \text{Im}((\mathcal{A} - \omega_1 - \omega_2)/\omega_3) > 0,$$

with $\mathcal{A} \equiv \sum_{n=0}^4 g_n$. Then the following integration formula holds

$$\int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{n=0}^4 G(g_n \pm u; \omega)}{G(\pm 2u, \mathcal{A} \pm u; \omega)} \frac{du}{\omega_2} = \kappa \frac{\prod_{0 \leq n < m \leq 4} G(g_n + g_m; \omega)}{\prod_{n=0}^4 G(\mathcal{A} - g_n; \omega)}, \quad (24a)$$

where

$$\kappa = \frac{-2(\tilde{q}; \tilde{q})_{\infty}}{(q; q)_{\infty} (p; p)_{\infty} (r; r)_{\infty}}. \quad (24b)$$

Here the integration is meant along the straight line segment connecting $-\omega_3/2$ to $\omega_3/2$ and we employed the shorthand notation $G(a \pm b; \omega) \equiv G(a + b, a - b; \omega)$.

Proof. We start by substituting relation (17a) into the left-hand side of (24a). This yields

$$e^{\pi i a} \int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{n=0}^4 \Gamma(e^{-2\pi i \frac{g_n \pm u}{\omega_3}}; \tilde{r}, \tilde{p})}{\Gamma(e^{\mp 4\pi i \frac{u}{\omega_3}}, e^{-2\pi i \frac{A \pm u}{\omega_3}}; \tilde{r}, \tilde{p})} \frac{du}{\omega_2}, \quad (25)$$

where

$$a = \frac{2}{3\omega_1\omega_2\omega_3} \left(\mathcal{A}^3 - \sum_{n=0}^4 g_n^3 \right) - \frac{\sum_{m=1}^3 \omega_m}{\omega_1\omega_2\omega_3} \left(\mathcal{A}^2 - \sum_{n=0}^4 g_n^2 \right) + \frac{1}{2} \left(\sum_{m=1}^3 \omega_m \right) \left(\sum_{m=1}^3 \frac{1}{\omega_m} \right).$$

The constraints on the parameters permit to employ formula (23) with the substitutions

$$z \rightarrow e^{\frac{2\pi i}{\omega_3} u}, \quad t_n \rightarrow e^{-\frac{2\pi i}{\omega_3} g_n}, \quad p \rightarrow e^{-2\pi i \frac{\omega_1}{\omega_3}}, \quad q \rightarrow e^{-2\pi i \frac{\omega_2}{\omega_3}},$$

which yields for (25)

$$\begin{aligned} & \frac{2\omega_3\omega_2^{-1}e^{\pi i a}}{(\tilde{r}; \tilde{r})_\infty(\tilde{p}; \tilde{p})_\infty} \frac{\prod_{0 \leq n < m \leq 4} \Gamma(e^{-2\pi i \frac{g_n + g_m}{\omega_3}}; \tilde{r}, \tilde{p})}{\prod_{n=0}^4 \Gamma(e^{-2\pi i \frac{A - g_n}{\omega_3}}; \tilde{r}, \tilde{p})} \\ &= \kappa \frac{\prod_{0 \leq n < m \leq 4} G(g_n + g_m; \omega)}{\prod_{n=0}^4 G(\mathcal{A} - g_n; \omega)}, \end{aligned}$$

with

$$\kappa = \frac{2\omega_3 e^{\frac{\pi i}{12} (\sum_{m=1}^3 \omega_m) (\sum_{m=1}^3 \omega_m^{-1})}}{\omega_2(\tilde{r}; \tilde{r})_\infty(\tilde{p}; \tilde{p})_\infty}.$$

After applying modular transformation (7) to the infinite products appearing in κ , we obtain

$$\kappa = -2 \sqrt{\frac{\omega_1}{i\omega_2}} \frac{e^{\frac{\pi i}{12} (\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})}}{(r; r)_\infty(p; p)_\infty}.$$

One more application of (7) allows us to replace the exponential function by a ratio of infinite products, which entails the desired form of κ in (24b). \square

Let us now consider the formal limit $p, r \rightarrow 0$ of the integral (24a). To this end, we fix the quasiperiods $\omega_{1,2}$ such that $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Re}(\omega_1/\omega_2) > 0$, and we furthermore pick $\omega_3 = i t \omega_2$ with $t > 0$. For $t \rightarrow +\infty$ the integral of Theorem 2 then degenerates formally to

$$\int_{\mathbb{L}} \frac{S(\pm 2u, \mathcal{A} \pm u; \omega)}{\prod_{n=0}^4 S(g_n \pm u; \omega)} \frac{du}{\omega_2} = -2 \frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} \frac{\prod_{n=0}^4 S(\mathcal{A} - g_n; \omega)}{\prod_{0 \leq n < m \leq 4} S(g_n + g_m; \omega)}, \quad (26)$$

where the integration is along the line $\mathbb{L} \equiv i\omega_2\mathbb{R}$, and with parameters subject to the constraints $\text{Re}(g_n/\omega_2) > 0$ and $\text{Re}((\mathcal{A} - \omega_1)/\omega_2) < 1$. This integral was deduced by a similar formal limit from the standard elliptic beta integral (23) and rigorously proved by Stokman in [St], where it was referred to as the hyperbolic Nassrallah-Rahman integral.

Remark 3. The generalized gamma function $[z; \tau]_\infty$ used in [St] coincides with the double sine function (16) upon setting $z = (\omega_2 - u)/\omega_2$ and $\tau = -\omega_2/\omega_1$.

Remark 4. To further elucidate the intimate relation between the elliptic beta integrals (23) and (24a), let us recall that—according to the general definition introduced in [S2]—a contour integral is called an *elliptic hypergeometric integral* if, for some displacement ω_1 , the ratio of its integrands $\Delta(u + \omega_1)/\Delta(u)$ constitutes an elliptic function of u with periods ω_2, ω_3 (say). Now, by the change of variables

$$z = e^{\frac{2\pi i}{\omega_2} u}, \quad t_n = e^{\frac{2\pi i}{\omega_2} g_n}, \quad A = e^{\frac{2\pi i}{\omega_2} \mathcal{A}},$$

the integral (23) passes into the additive form

$$\int_{-\omega_2/2}^{\omega_2/2} \Delta(u) \frac{du}{\omega_2} = \frac{2 \prod_{0 \leq n < m \leq 4} \Gamma(e^{\frac{2\pi i}{\omega_2} \frac{g_n + g_m}{2}}; q, p)}{(q; q)_\infty (p; p)_\infty \prod_{n=0}^4 \Gamma(e^{\frac{2\pi i}{\omega_2} \frac{\mathcal{A} - g_n}{2}}; q, p)}, \quad (27a)$$

with the integrand given by

$$\Delta(u) = \frac{\prod_{n=0}^4 \Gamma(e^{\frac{2\pi i}{\omega_2} \frac{g_n \pm u}{2}}; q, p)}{\Gamma(e^{\pm 4\pi i \frac{u}{\omega_2}}, e^{\frac{2\pi i}{\omega_2} \frac{\mathcal{A} \pm u}{2}}; q, p)}, \quad (27b)$$

where $\Gamma(e^{a \pm b}; q, p) \equiv \Gamma(e^{a+b}, e^{a-b}; q, p)$. We then have that

$$\frac{\Delta(u + \omega_1)}{\Delta(u)} = e^{2\pi i \frac{\omega_1}{\omega_2}} \frac{\theta(e^{4\pi i \frac{u + \omega_1}{\omega_2}}; p) \theta(e^{2\pi i \frac{u + \omega_1 - \mathcal{A}}{\omega_2}}; p)}{\theta(e^{4\pi i \frac{u}{\omega_2}}; p) \theta(e^{2\pi i \frac{u + \mathcal{A}}{\omega_2}}; p)} \prod_{n=0}^4 \frac{\theta(e^{2\pi i \frac{u + g_n}{\omega_2}}; p)}{\theta(e^{2\pi i \frac{u + \omega_1 - g_n}{\omega_2}}; p)}, \quad (28)$$

which is seen to be an elliptic function of u with the periods ω_2 and ω_3 . With the aid of the difference equation (14a) for the modified gamma function, it is not difficult to verify that the integrand of modified elliptic beta integral (24a) also provides a solution to (28). Hence both integrals (23) and (24a) are elliptic hypergeometric integrals with integrands satisfying (28). Whereas the solution of (28) originating from integral (23) is well-defined only for $|q| < 1$ (or for $|q| > 1$ upon performing the inversion $q \rightarrow q^{-1}$ in the elliptic gamma function [S2]), the modified integral (24a) corresponds to a solution that extends analytically from the regime $|q| < 1$ to the unit circle $|q| = 1$. In this sense the modified integral (24a) may be seen as a $|q| = 1$ analog of the original elliptic beta integral (23).

5. MULTIPLE INTEGRALS

The following integral constitutes a multidimensional generalization of the elliptic beta integral in (23)

$$\begin{aligned} & \frac{1}{(2\pi i)^N} \int_{\mathbb{T}^N} \prod_{1 \leq j < k \leq N} \frac{\Gamma(tz_j^\pm z_k^\pm; q, p)}{\Gamma(z_j^\pm z_k^\pm; q, p)} \prod_{j=1}^N \frac{\prod_{n=0}^4 \Gamma(t_n z_j^\pm; q, p)}{\Gamma(z_j^{\pm 2}, B z_j^\pm; q, p)} \frac{dz_1}{z_1} \dots \frac{dz_N}{z_N} \\ &= \frac{2^N N!}{(p; p)_\infty^N (q; q)_\infty^N} \prod_{j=1}^N \frac{\Gamma(t^j; q, p)}{\Gamma(t; q, p)} \frac{\prod_{0 \leq n < m \leq 4} \Gamma(t^{j-1} t_n t_m; q, p)}{\prod_{n=0}^4 \Gamma(t^{1-j} t_n^{-1} B; q, p)}, \end{aligned} \quad (29)$$

where $B \equiv t^{2N-2} \prod_{n=0}^4 t_n$, $\Gamma(tz^\pm x^\pm; q, p) \equiv \Gamma(tzx, tzx^{-1}, tz^{-1}x, tz^{-1}x^{-1}; q, p)$, and with parameters subject to the constraints $|p|, |q|, |t|, |t_n| < 1$ and $|pq| < |B|$. This multiple elliptic beta integral was first formulated as a conjecture in [DS1]. Next, it was shown in [DS2] that the conjecture in question follows from a vanishing hypothesis for a related multiparameter elliptic beta integral. Recently, a complete proof of the integral in (29) was found by Rains [R].

The following theorem provides the corresponding multidimensional generalization of the modified elliptic beta integral in Theorem 2.

Theorem 3. *Let $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$, and let $g, g_n, n = 0, \dots, 4$, be six complex parameters subject to the constraints*

$$\text{Im}(g/\omega_3), \text{Im}(g_n/\omega_3) < 0, \quad \text{Im}((\mathcal{B} - \omega_1 - \omega_2)/\omega_3) > 0,$$

with $\mathcal{B} \equiv (2N - 2)g + \sum_{n=0}^4 g_n$. Then

$$\begin{aligned} & \int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}} \cdots \int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}} \prod_{1 \leq j < k \leq N} \frac{G(g \pm u_j \pm u_k; \omega)}{G(\pm u_j \pm u_k; \omega)} \prod_{j=1}^N \frac{\prod_{n=0}^4 G(g_n \pm u_j; \omega)}{G(\pm 2u_j, \mathcal{B} \pm u_j; \omega)} \frac{du_1}{\omega_2} \cdots \frac{du_N}{\omega_2} \\ &= \kappa^N N! \prod_{j=1}^N \frac{G(jg; \omega)}{G(g; \omega)} \frac{\prod_{0 \leq n < m \leq 4} G((j-1)g + g_n + g_m; \omega)}{\prod_{n=0}^4 G((1-j)g + \mathcal{B} - g_n; \omega)}, \end{aligned} \quad (30)$$

with κ given by (24b) and $G(c \pm a \pm b; \omega) \equiv G(c+a+b, c+a-b, c-a+b, c-a-b; \omega)$.

Proof. The proof is analogous to that of Theorem 2. Specifically, after substituting (17a) into the left-hand side of (30) and application of the multiple beta integral in (29), we arrive at the integration formula stated in the theorem upon expressing the resulting evaluation in terms of the modified gamma function $G(u; \omega)$. To infer the correctness of the value of the proportionality constant $\kappa^N N!$, one observes that the dependence on g in the factors originating from the exponential multipliers cancels out. The expression for the proportionality constant then follows from the fact that for $g \rightarrow 0$ integral (30) reduces to the N -th power of the elliptic beta integral (24a). \square

Remark 5. In [DS2, S2] various other types of multiple elliptic beta integrals were formulated. These can be extended to the unit circle $|q| = 1$ in a similar fashion.

For $p = 0$, elliptic beta integral (29) reduces to a Gustafson's multiple integral corresponding to the Nassrallah-Rahman type generalization of the Selberg integral [Gu2]. The following theorem provides a corresponding multiple analog of the integral in (26). The integration formula in question can be obtained formally from the modified elliptic beta integral (30) with $\omega_1/\omega_2 > 0$ by taking the limit $p, r \rightarrow 0$ in the manner explained below Theorem 2.

Theorem 4. *Let ω_1, ω_2 be quasiperiods such that $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Re}(\omega_1/\omega_2) > 0$. Furthermore, let $g, g_n, n = 0, \dots, 4$, be parameters subject to the constraints $\text{Re}(g/\omega_1), \text{Re}(g/\omega_2), \text{Re}(g_n/\omega_2) > 0$ and $\text{Re}((\mathcal{B} - \omega_1)/\omega_2) < 1$ (with \mathcal{B} as in Theorem 3). Then*

$$\int_{\mathbb{L}^N} \Delta(\mathbf{u}; \mathbf{g}) \frac{du_1}{\omega_2} \cdots \frac{du_N}{\omega_2} = \mathcal{N}(\mathbf{g}), \quad (31a)$$

where $\mathbb{L} = i\omega_2\mathbb{R}$,

$$\Delta(\mathbf{u}; \mathbf{g}) = \prod_{1 \leq j < k \leq N} \frac{S(\pm u_j \pm u_k; \omega)}{S(g \pm u_j \pm u_k; \omega)} \prod_{j=1}^N \frac{S(\pm 2u_j, \mathcal{B} \pm u_j; \omega)}{\prod_{n=0}^4 S(g_n \pm u_j; \omega)} \quad (31b)$$

and

$$\mathcal{N}(\mathbf{g}) = (-2)^N N! \frac{(\tilde{q}; \tilde{q})_\infty^N}{(q; q)_\infty^N} \prod_{j=1}^N \frac{S(jg; \omega)}{S(g; \omega)} \frac{\prod_{n=0}^4 S((1-j)g + \mathcal{B} - g_n; \omega)}{\prod_{0 \leq n < m \leq 4} S((j-1)g + g_n + g_m; \omega)}. \quad (31c)$$

Through a parameter specialization, Gustafson's multiple integral of the Nassrallah-Rahman type can be reduced to a multiple Askey-Wilson integral first derived in [Gu1]. The corresponding degeneration of Theorem 4 reads as follows.

Theorem 5. *Let ω_1, ω_2 be quasiperiods such that $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Re}(\omega_1/\omega_2) > 0$, and let $g, g_n, n = 0, \dots, 3$, be parameters subject to the constraints $\text{Re}(g/\omega_1), \text{Re}(g/\omega_2), \text{Re}(g_n/\omega_2) > 0$ and $\text{Re}((\mathcal{B} - \omega_2)/\omega_1) < 1$ with $\mathcal{B} \equiv (2N - 2)g + \sum_{n=0}^3 g_n$. Then*

$$\begin{aligned} & \int_{\mathbb{L}^N} \prod_{1 \leq j < k \leq N} \frac{S(\pm u_j \pm u_k; \omega)}{S(g \pm u_j \pm u_k; \omega)} \prod_{j=1}^N \frac{S(\pm 2u_j; \omega)}{\prod_{n=0}^3 S(g_n \pm u_j; \omega)} \frac{du_1}{\omega_2} \dots \frac{du_N}{\omega_2} \\ &= (-2)^N N! \frac{(\tilde{q}; \tilde{q})_\infty^N}{(q; q)_\infty^N} \prod_{j=1}^N \frac{S(g; \omega)}{S(jg; \omega)} \frac{S((1-j)g + \mathcal{B}; \omega)}{\prod_{0 \leq n < m \leq 3} S((j-1)g + g_n + g_m; \omega)}. \end{aligned} \quad (32)$$

For $N = 1$, the integral in Theorem 5 reduces to a single variable Askey-Wilson type integral

$$\int_{\mathbb{L}} \frac{S(\pm 2u; \omega)}{\prod_{n=0}^3 S(g_n \pm u; \omega)} \frac{du}{\omega_2} = -2 \frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} \frac{S(g_0 + g_1 + g_2 + g_3; \omega)}{\prod_{0 \leq n < m \leq 3} S(g_n + g_m; \omega)}, \quad (33)$$

which was established by Ruijsenaars [Ru3] and Stokman [St].

Formally, the integral of Theorem 5 follows from that of Theorem 4 with $\omega_1/\omega_2 > 0$ upon setting $g_4 \rightarrow g_4 + i\omega_2 t$ and performing the limit $t \rightarrow +\infty$. However, it is not a simple matter to upgrade such formal limiting relations between Theorem 3 and Theorems 4, 5 to rigorous proofs of the latter integration formulas. A direct proof of Theorems 4 and 5, modelled after Rains' proof of the multiple elliptic beta integral (29), is given in Section 6 below. As it was communicated to us by Stokman after finishing this paper, Theorem 5 can be proved by a multivariable generalization of the method of [St] as well.

6. PROOF OF THEOREMS 4 AND 5

We first detail the proof of Theorem 4 and then indicate some modifications so as to incorporate Theorem 5.

Let us for the moment assume that the quasiperiods ω_1, ω_2 are incommensurate over \mathbb{Q} . The double sine function $S(u; \omega)$ then has simple zeros located at the points $u = -\omega_1 \mathbb{N} - \omega_2 \mathbb{N}$ and simple poles at $u = \omega_1(1 + \mathbb{N}) + \omega_2(1 + \mathbb{N})$. Therefore, the integrand $\Delta(\mathbf{u}; \mathbf{g})$ in (31a) has poles at the points

$$\begin{aligned} \pm u_j &= -\mathcal{B} + \omega_1(1 + \mathbb{N}) + \omega_2(1 + \mathbb{N}), \quad g_n + \omega_1 \mathbb{N} + \omega_2 \mathbb{N}, \quad n = 0, \dots, 4, \\ g + u_k + \omega_1 \mathbb{N} + \omega_2 \mathbb{N}, \quad g - u_k + \omega_1 \mathbb{N} + \omega_2 \mathbb{N}, \quad k = 1, \dots, N, \quad k \neq j, \end{aligned}$$

where $j = 1, \dots, N$.

Combination with the asymptotics in Eqs. (21a), (21b) reveals that the quotients $S(u; \omega)/S(g + u; \omega)$ and $S(2u, \mathcal{B} + u; \omega)/\prod_{n=0}^4 S(g_n + u)$ are smooth and bounded on the complex line $\mathbb{L} = i\omega_2 \mathbb{R}$. Indeed, for $u = ix\omega_2, x \in \mathbb{R}$ we stay away from poles and we have that

$$\frac{S(i\omega_2 x; \omega)}{S(g + i\omega_2 x; \omega)} = \begin{cases} O(1) & \text{for } x \rightarrow +\infty \\ O(e^{-2\pi x g/\omega_1}) & \text{for } x \rightarrow -\infty \end{cases}$$

and

$$\frac{S(2i\omega_2 x, \mathcal{B} + i\omega_2 x; \boldsymbol{\omega})}{\prod_{n=0}^4 S(g_n + i\omega_2 x; \boldsymbol{\omega})} = \begin{cases} O(1) & \text{for } x \rightarrow +\infty \\ O(e^{2\pi x(2(N-1)g/\omega_1 + 1 + \omega_2/\omega_1)}) & \text{for } x \rightarrow -\infty \end{cases}.$$

It thus follows that the integrand $\Delta(\mathbf{u}; \mathbf{g})$ is smooth and exponentially decaying at infinity on the integration domain \mathbb{L}^N . Hence, the integral in Eq. (31a) converges.

To infer the validity of the integration formula we distinguish three parameters g_0, g_1, g_2 and factorize the integrand as $\Delta(\mathbf{u}; \mathbf{g}) = \Delta_+(\mathbf{u})\Delta_-(\mathbf{u})$ with

$$\Delta_+(\mathbf{u}) = \prod_{1 \leq j < k \leq N} \frac{S(u_j \pm u_k; \boldsymbol{\omega})}{S(g + u_j \pm u_k; \boldsymbol{\omega})} \prod_{j=1}^N \frac{S(2u_j, \mathcal{B} - u_j, \omega_1 + \mathcal{C} - u_j; \boldsymbol{\omega})}{S(\omega_1 + \mathcal{C} + u_j; \boldsymbol{\omega}) \prod_{n=0}^4 S(g_n + u_j; \boldsymbol{\omega})}, \quad (34)$$

where $\mathcal{C} = (N-1)g + g_0 + g_1 + g_2$ and $\Delta_-(\mathbf{u}) = \Delta_+(-u_1, \dots, -u_N)$. Similarly, we introduce the shifted functions

$$\begin{aligned} \tilde{\Delta}_+(\mathbf{u}) &= \prod_{1 \leq j < k \leq N} \frac{S(u_j \pm u_k; \boldsymbol{\omega})}{S(g + u_j \pm u_k; \boldsymbol{\omega})} \\ &\times \prod_{j=1}^N \frac{S(2u_j, \mathcal{B} + \frac{\omega_1}{2} - u_j, \mathcal{C} + \frac{\omega_1}{2} - u_j; \boldsymbol{\omega})}{S(\mathcal{C} + \frac{\omega_1}{2} + u_j, g_3 - \frac{\omega_1}{2} + u_j, g_4 - \frac{\omega_1}{2} + u_j; \boldsymbol{\omega}) \prod_{n=0}^2 S(g_n + \frac{\omega_1}{2} + u_j; \boldsymbol{\omega})}, \end{aligned} \quad (35)$$

and $\tilde{\Delta}_-(\mathbf{u}) = \tilde{\Delta}_+(-u_1, \dots, -u_N)$, which provide a factorization of the integrand for a shifted set of parameter values:

$$\tilde{\Delta}_+(\mathbf{u})\tilde{\Delta}_-(\mathbf{u}) = \Delta(\mathbf{u}; g, g_0 + \frac{\omega_1}{2}, g_1 + \frac{\omega_1}{2}, g_2 + \frac{\omega_1}{2}, g_3 - \frac{\omega_1}{2}, g_4 - \frac{\omega_1}{2}).$$

Let us for the moment assume that the parameters are such that the shifted parameters (as well as the ones obtained after shifts by $\pm\omega_2/2$) also belong to the domain indicated in the theorem. We then have the following equality

$$\begin{aligned} &\int_{\mathbb{L}^N} \tilde{\Delta}_+(u_1 + \frac{\omega_1}{2}, \dots, u_N + \frac{\omega_1}{2}) \Delta_-(\mathbf{u}) du_1 \cdots du_N \\ &= \int_{\mathbb{L}^N} \tilde{\Delta}_+(\mathbf{u}) \Delta_-(u_1 - \frac{\omega_1}{2}, \dots, u_N - \frac{\omega_1}{2}) du_1 \cdots du_N, \end{aligned} \quad (36)$$

which follows by shifting the integration contours \mathbb{L} on the left-hand side by $-\omega_1/2$. Notice that such shifts are permitted by the Cauchy theorem due to the absence of poles in the strip between \mathbb{L} and $\mathbb{L} - \omega_1/2$ combined with the exponential decay at infinity. Indeed, the quotient $S(u; \boldsymbol{\omega})/S(g + u; \boldsymbol{\omega})$ is holomorphic on the strip $\{u = s\omega_1 + ix\omega_2 \mid 0 \leq s \leq 1, -\infty < x < \infty\}$ and the quotient $S(2u, \mathcal{B} + u, \mathcal{C} + u; \boldsymbol{\omega}) / (S(\mathcal{C} + \frac{\omega_1}{2} + u, g_3 - \frac{\omega_1}{2} + u, g_4 - \frac{\omega_1}{2} + u; \boldsymbol{\omega}) \prod_{n=0}^2 S(g_n + \frac{\omega_1}{2} + u; \boldsymbol{\omega}))$ is holomorphic on the strip $\{u = s\omega_1 + ix\omega_2 \mid 0 \leq s \leq 1/2, -\infty < x < \infty\}$. By performing sign flips of the form $u_j \rightarrow -u_j$ in the integration variables and summing over all 2^N possible ways, we obtain from (36) that

$$\begin{aligned} &\int_{\mathbb{L}^N} \rho(\mathbf{u}; \mathbf{g}) \Delta_+(\mathbf{u}) \Delta_-(\mathbf{u}) du_1 \cdots du_N \\ &= \int_{\mathbb{L}^N} \tilde{\rho}(\mathbf{u}; \mathbf{g}) \tilde{\Delta}_+(\mathbf{u}) \tilde{\Delta}_-(\mathbf{u}) du_1 \cdots du_N, \end{aligned} \quad (37)$$

with

$$\begin{aligned}\rho(\mathbf{u}; \mathbf{g}) &= \sum_{\substack{\nu_\ell = \pm 1 \\ \ell=1, \dots, N}} \frac{\tilde{\Delta}_+(\nu_1 u_1 + \frac{\omega_1}{2}, \dots, \nu_N u_N + \frac{\omega_1}{2})}{\Delta_+(\nu_1 u_1, \dots, \nu_N u_N)}, \\ \tilde{\rho}(\mathbf{u}; \mathbf{g}) &= \sum_{\substack{\nu_\ell = \pm 1 \\ \ell=1, \dots, N}} \frac{\Delta_-(\nu_1 u_1 - \frac{\omega_1}{2}, \dots, \nu_N u_N - \frac{\omega_1}{2})}{\tilde{\Delta}_-(\nu_1 u_1, \dots, \nu_N u_N)}.\end{aligned}$$

Simplification of $\rho(\mathbf{u}; \mathbf{g})$ reveals that

$$\begin{aligned}\rho(\mathbf{u}; \mathbf{g}) &= \sum_{\substack{\nu_\ell = \pm 1 \\ \ell=1, \dots, N}} \prod_{1 \leq j < k \leq N} \frac{1 - t z_j^{\nu_j} z_k^{\nu_k}}{1 - z_j^{\nu_j} z_k^{\nu_k}} \prod_{j=1}^N \frac{(1 - t^{N-1} t_0 t_1 t_2 z_j^{-\nu_j}) \prod_{n=0}^2 (1 - t_n z_j^{\nu_j})}{1 - z_j^{2\nu_j}} \\ &= \prod_{j=1}^N (1 - t^{j-1} t_0 t_1) (1 - t^{j-1} t_0 t_2) (1 - t^{j-1} t_1 t_2),\end{aligned}\tag{38}$$

with $t = e^{2\pi i g / \omega_2}$, $t_n = e^{2\pi i g_n / \omega_2}$, $z_k = e^{2\pi i u_k / \omega_2}$. The summand in (38) is invariant under permutations of z_k and inversions $z_k \rightarrow z_k^{-1}$. The product of $\rho(\mathbf{u}; \mathbf{g})$ and the factor

$$\prod_{1 \leq j < k \leq N} \frac{(1 - z_j z_k)(1 - z_j z_k^{-1})}{z_j} \prod_{j=1}^N \frac{1 - z_j^2}{z_j}$$

yields a Laurent polynomial in z_j , $j = 1, \dots, N$, which is antisymmetric with respect to both transformations (separate permutations of z_j and inversions $z_j \rightarrow z_j^{-1}$). Any such polynomial is proportional to the multiplicative factor given above. The constant of proportionality is found after setting $z_j = t_0 t^{N-j}$, which leaves only one term in the sum (with all $\nu_j = 1$) giving the right-hand side expression in (38).

In the same way, one obtains for $\tilde{\rho}(\mathbf{u}; \mathbf{g})$ that replacing in relation (38) $t_{0,1,2}$ by $t_{3,4} q^{-1/2}$, $t^{N-1} t_0 t_1 t_2 q^{1/2}$,

$$\begin{aligned}\tilde{\rho}(\mathbf{u}; \mathbf{g}) &= \sum_{\substack{\nu_\ell = \pm 1 \\ \ell=1, \dots, N}} \prod_{1 \leq j < k \leq N} \frac{1 - t z_j^{\nu_j} z_k^{\nu_k}}{1 - z_j^{\nu_j} z_k^{\nu_k}} \prod_{j=1}^N \left(\frac{(1 - t_3 q^{-1/2} z_j^{\nu_j})(1 - t_4 q^{-1/2} z_j^{\nu_j})}{1 - z_j^{2\nu_j}} \right. \\ &\quad \left. \times (1 - t^{N-1} t_0 t_1 t_2 q^{1/2} z_j^{\nu_j})(1 - B q^{-1/2} z_j^{-\nu_j}) \right) \\ &= \prod_{j=1}^N (1 - t^{j-1} t_3 t_4 / q) (1 - t^{1-j} B / t_3) (1 - t^{1-j} B / t_4),\end{aligned}$$

where $B = e^{2\pi i \mathcal{B} / \omega_2}$. The derived expressions demonstrate that the functions in question are constant in the integration variables \mathbf{u} , that is $\rho(\mathbf{u}; \mathbf{g}) = \rho(\mathbf{g})$ and $\tilde{\rho}(\mathbf{u}; \mathbf{g}) = \tilde{\rho}(\mathbf{g})$. Hence, we can pull the corresponding factors out of the integrals and rewrite (37) as

$$\int_{\mathbb{L}^N} \Delta_+(\mathbf{u}) \Delta_-(\mathbf{u}) du_1 \cdots du_N = \frac{\tilde{\rho}(\mathbf{g})}{\rho(\mathbf{g})} \int_{\mathbb{L}^N} \tilde{\Delta}_+(\mathbf{u}) \tilde{\Delta}_-(\mathbf{u}) du_1 \cdots du_N,$$

whence

$$\frac{\mathcal{N}(\mathbf{g})}{\mathcal{N}(g, g_0 + \frac{\omega_1}{2}, g_1 + \frac{\omega_1}{2}, g_2 + \frac{\omega_1}{2}, g_3 - \frac{\omega_1}{2}, g_4 - \frac{\omega_1}{2})} = \frac{\tilde{\rho}(\mathbf{g})}{\rho(\mathbf{g})}.\tag{39}$$

As a result, we deduce that the ratio of the left- and right-hand sides of (31a) is invariant with respect to the shifts $g_{0,1,2} \rightarrow g_{0,1,2} + \omega_1/2$, $g_{3,4} \rightarrow g_{3,4} - \omega_1/2$, and, by symmetry, any permutation of indices of the parameters. The double sine function $S(u; \omega)$ and, so, the integrand in (31a) and the integral's value $\mathcal{N}(\mathbf{g})$ are symmetric with respect to the permutation of ω_1 and ω_2 [KLS]. The contour of integration \mathbb{L} breaks the symmetry between $\omega_{1,2}$, but the transformations used in (34)-(39) are purely algebraic and do not depend on the contour of integration. Therefore, the ratio of interest is invariant with respect to the shifts $g_{0,1,2} \rightarrow g_{0,1,2} + \omega_2/2$, $g_{3,4} \rightarrow g_{3,4} - \omega_2/2$ as well (with all permutations of indices of parameters).

By analyticity, without changing the integral's value we can replace the contour of integration \mathbb{L} by any other contour embracing the same set of poles. For an appropriately deformed contour, we can establish invariance of the ratio of interest under the shifts $g_n \rightarrow g_n + k\omega_1/2 + m\omega_2/2$, for arbitrary $k, m \in \mathbb{Z}$. Taking $\omega_1, \omega_2 > 0$, we can choose a subset of these points with a limiting point in the parameter space for which we can choose \mathbb{L} as the integration contour. Moreover, making sequential $\omega_{1,2}/2$ shifts in different directions we can escape large intermediate deformations of the integration contour (such an argument is similar to the one given in [S1]). Therefore, the ratio of the left- and right-hand sides of (31a) is equal to a function of $\omega_{1,2}$ and g , which we denote as $f(\omega_1, \omega_2, g)$.

In order to see that $f(\omega_1, \omega_2, g)$ actually equals to one, it is necessary to use an analog of the residue formula derived in [DS1]. Namely, we take one of the parameters, say, g_0 such that one pole from the half plane $\text{Re}(u/\omega_2) > 0$ crosses the contour of integration \mathbb{L} . A similar move takes place for the pole at $u = -g_0$ from the $\text{Re}(u/\omega_2) < 0$ half plane (due to the reflection invariance). We keep intact all other poles in the half planes to the left or right of \mathbb{L} . This is possible to do by an appropriate choice of g, g_1, \dots, g_4 , and $\omega_{1,2}$. It is not difficult to see that the residues of these crossing poles taken, say, over the variable u_N have poles at the points $u_k = \pm(g_0 + g), k = 1, \dots, N-1$, (instead of $u_k = \pm g_0$) and they are still located to the left or right of \mathbb{L} for $\text{Re}((g_0 + g)/\omega_2) < 0$. Similar shifts of the poles occur each time we calculate the residues. Therefore we denote $\rho_k = g_0 + (k-1)g$ and impose the restrictions

$$\text{Re}\left(\frac{\rho_k}{\omega_2}\right) < 0, \quad k = 1, \dots, N, \quad \text{Re}\left(\frac{g_0 + \omega_1}{\omega_2}\right), \text{Re}\left(\frac{g_0 + \omega_2}{\omega_2}\right) > 0.$$

Because of the taken constraints upon $g, \omega_{1,2}$, we get a simpler residue formula than the one derived in [DS1]:

$$\begin{aligned} & \int_{\mathbb{L}_d^N} \Delta(\mathbf{u}; \mathbf{g}) \frac{du_1}{\omega_2} \dots \frac{du_N}{\omega_2} \\ &= \sum_{m=0}^N 2^m m! \binom{N}{m} \int_{\mathbb{L}^{N-m}} \mu_m(\mathbf{u}) \frac{du_1}{\omega_2} \dots \frac{du_{N-m}}{\omega_2}, \end{aligned} \quad (40)$$

where \mathbb{L}_d is a deformation of the contour \mathbb{L} such that it separates the same sets of poles as \mathbb{L} did before we started to change g_0 . The factor 2^m emerges because the residues appear in pairs and their values coincide (due to the $u_k \rightarrow -u_k$ reflection invariance of the integrand and different orientation of the contours encircling poles to the left and right of \mathbb{L}). The factors $m!$ and $\binom{N}{m}$ count the number of orderings of m cycles and the number of ways to pick up these cycles out of N possibilities.

The residue functions have the form $\mu_0(\mathbf{u}) = \Delta(\mathbf{u}; \mathbf{g})$ and for $m > 0$

$$\mu_m(\mathbf{u}) = \kappa_m \delta_{m, N-m}(\mathbf{u}) \Delta_{N-m}(\mathbf{u}; \mathbf{g}), \quad (41)$$

where $\Delta_{N-m}(\mathbf{u}; \mathbf{g})$ is obtained from integrand (31b) if we replace in it N by $N-m$ but keep $\mathcal{B} = (2N-2)g + \sum_{n=0}^4 g_n$ unchanged. Other coefficients are

$$\kappa_m = (-1)^m \frac{(\tilde{q}; \tilde{q})_\infty^m}{(q; q)_\infty^m} \prod_{1 \leq j < k \leq m} \frac{S(\pm \rho_k - \rho_j; \boldsymbol{\omega})}{S(g \pm \rho_k - \rho_j; \boldsymbol{\omega})} \prod_{l=1}^m \frac{S(-2\rho_l, \mathcal{B} \pm \rho_l; \boldsymbol{\omega})}{\prod_{n=1}^4 S(g_n \pm \rho_l; \boldsymbol{\omega})},$$

and

$$\delta_{m, N-m}(\mathbf{u}) = \prod_{\substack{1 \leq j \leq m \\ 1 \leq k \leq N-m}} \frac{S(\pm \rho_j \pm u_k; \boldsymbol{\omega})}{S(g \pm \rho_j \pm u_k; \boldsymbol{\omega})}.$$

The expressions for $\mu_m(\mathbf{u})$ are derived by induction. Indeed, the form of $\mu_1(\mathbf{u})$ is easily established after taking into account the relation

$$\lim_{u \rightarrow \pm g_0} \frac{u \mp g_0}{S(g_0 \mp u; \boldsymbol{\omega})} = \pm \frac{\omega_2}{2\pi i} \frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty}$$

and the fact that the contours encircling the corresponding poles are oriented clockwise for the upper signs and anticlockwise for the lower signs (this gives the total minus sign in κ_1).

Suppose that μ_m is given by (41) for some $m > 1$. In order to find μ_{m+1} it is necessary to compute the residues for poles located at $u_{N-m} = \pm \rho_{m+1}$. A simple computation shows that, indeed,

$$\int_{c_m} \mu_m(\mathbf{u}) \frac{du_{N-m}}{\omega_2} = \mu_{m+1}(\mathbf{u}),$$

where c_m is a small size clockwise orientated contour encircling the pole at $u_{N-m} = \rho_{m+1}$.

By analyticity, our deformations of the parameters and of the contour of integration do not change the integral value and, therefore, the right-hand side sum in (40) equals to $f(\omega_1, \omega_2, g) \mathcal{N}(\mathbf{g})$. We now divide both sides of this equality by $\mathcal{N}(\mathbf{g})$ and take the limit $g_4 \rightarrow -g_0 - (N-1)g$. For $m < N$, the coefficients $\kappa_m(\mathbf{g})$, which can be represented in the form

$$\begin{aligned} \kappa_m = (-1)^m \frac{(\tilde{q}; \tilde{q})_\infty^m}{(q; q)_\infty^m} \prod_{l=1}^m & \left(\frac{S(g, (2-m-l)g - 2g_0; \boldsymbol{\omega})}{S(lg; \boldsymbol{\omega})} \right. \\ & \times \left. \frac{S(2g_0 + \sum_{r=1}^4 g_r + (2n+l-3)g, \sum_{r=1}^4 g_r + (2n-l-1)g; \boldsymbol{\omega})}{\prod_{r=1}^4 S(g_r + g_0 + (l-1)g, g_r - g_0 - (l-1)g; \boldsymbol{\omega})} \right), \end{aligned}$$

do not contain diverging factors in this limit and the integrals, which they are multiplied by, remain bounded. Therefore, only the term with $m = N$ survives and, by simple computation, we obtain

$$\lim_{g_4 \rightarrow -g_0 - (N-1)g} \frac{2^N N! \kappa_N}{\mathcal{N}(\mathbf{g})} = 1,$$

which means that $f(\omega_1, \omega_2, g) = 1$. After proving equality (31a) in the taken restricted region of parameters (where the parameters shifted by $\pm \omega_{1,2}/2$ satisfy the needed constraints and $\omega_{1,2} > 0$), we can analytically extend it to the values

of $\omega_{1,2}$ and parameters g, g_n in the domain indicated in the formulation of the theorem. Theorem 4 is thus proved.

We now turn to the Askey-Wilson type integral (31a). Its convergence conditions essentially differ from the previous case. Indeed, we have

$$\frac{S(2i\omega_2 x; \omega)}{\prod_{n=0}^3 S(g_n + i\omega_2 x; \omega)} = \begin{cases} O(1) & \text{for } x \rightarrow +\infty \\ O(e^{2\pi x(1+\omega_2/\omega_1 - \sum_{n=0}^3 g_n/\omega_1)}) & \text{for } x \rightarrow -\infty \end{cases}.$$

Combining together these limiting relations with the asymptotics for the ratio $S(i\omega_2 x; \omega)/S(g + i\omega_2 x; \omega)$, we see that the integrand remains bounded in the integration domain \mathbb{L}^N and decays exponentially fast on its infinities if we take $\text{Re}((\mathcal{B} - \omega_2)/\omega_1) < 1$, where $\mathcal{B} = (2N - 2)g + \sum_{n=0}^3 g_n$.

Invariance of the ratio of left- and right-hand sides of equality (31a) under the specified parameter shifts relied only on algebraic manipulations with the integrand. Therefore we can repeat them for the limiting expression of the integrand appearing after taking the limit $\text{Im}(g_4/\omega_2), \text{Im}(g_4/\omega_1) \rightarrow +\infty$ (or $t_4 \rightarrow 0$). This simplifies the integrand for (31a) to the one for (32). Therefore, limiting analogs of equalities (34)–(39) show that the ratio of the left- and right-hand sides of (32) do not depend on the shifts in the parameter space $g_{0,1,2} \rightarrow g_{0,1,2} + \omega_{1,2}/2$, $g_3 \rightarrow g_3 - \omega_{1,2}/2$ and the ones obtained by permutation of indices. Using, again, an analytical continuation and the appropriately simplified version of the residue calculus, we see that the ratio of interest is actually equal to one. As a result, we establish validity of integral (32) as well.

Acknowledgments

The second author thanks the Instituto de Matemática y Física of the Universidad de Talca for the hospitality during the visit in May 2003, at which time the main results of this paper were obtained. This work is supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grants No. # 1010217 and No. # 7010217, by the Programa Formas Cuadráticas of the Universidad de Talca, and by the Russian Foundation for Basic Research (RFBR) Grant No. 03-01-00781.

REFERENCES

- AAR. G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Math. Appl. **71**, Cambridge Univ. Press, Cambridge, 1999.
- Ba. E. W. Barnes, *On the theory of the multiple gamma function*, Trans. Cambridge Phil. Soc. **19** (1904), 374–425.
- DS1. J. F. van Diejen and V. P. Spiridonov, *An elliptic Macdonald-Morris conjecture and multiple modular hypergeometric sums*, Math. Res. Letters **7** (2000), 729–746.
- DS2. ———, *Elliptic Selberg integrals*, Internat. Math. Res. Notices, no. 20 (2001), 1083–1110.
- F. L. D. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Lett. Math. Phys. **34** (1995), 249–254; *Modular double of a quantum group*, Conf. Moshé Flato 1999, vol. I, Math. Phys. Stud. **21**, Kluwer, Dordrecht, 2000, pp. 149–156.
- FKV. L. D. Faddeev, R. M. Kashaev, and A. Yu. Volkov, *Strongly coupled quantum discrete Liouville Theory. I: Algebraic approach and duality*, Commun. Math. Phys. **219** (2001), 199–219.
- FV. G. Felder and A. Varchenko, *The elliptic gamma function and $SL(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$* , Adv. Math. **156** (2000), 44–76.
- Gu1. R. A. Gustafson, *A generalization of Selberg’s beta integral*, Bull. Am. Math. Soc., New Ser. **22** (1990), 97–105.
- Gu2. ———, *Some q -beta integrals on $SU(n)$ and $Sp(n)$ that generalize the Askey-Wilson and Nassrallah-Rahman integrals*, SIAM J. Math. Anal. **25** (1994), 441–449.

- J. F. H. Jackson, *The basic gamma-function and the elliptic functions*, Proc. Roy. Soc. London **A76** (1905), 127–144.
- JM. M. Jimbo and T. Miwa, *Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime*, J. Phys. A: Math. Gen. **29** (1996), 2923–2958.
- KLS. S. Kharchev, D. Lebedev, and M. Semenov-Tian-Shansky, *Unitary representations of $U_q(sl(2, \mathbb{R}))$, the modular double and the multiparticle q -deformed Toda chains*, Commun. Math. Phys. **225** (2002), 573–609.
- Ku. N. Kurokawa, *Multiple sine functions and Selberg zeta functions*, Proc. Japan Acad. **67A** (1991), 61–64.
- M. Yu. Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*, Astérisque **228** (4) (1995), 121–163.
- NU. M. Nishizawa and K. Ueno, *Integral solutions of hypergeometric q -difference systems with $|q| = 1$* , Physics and Combinatorics (Nagoya, 1999), World Scientific, River Edge, 2001, pp. 273–286.
- PT. B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U_q(sl(2, \mathbb{R}))$* , Commun. Math. Phys. **224** (2001), 613–655.
- Rah. M. Rahman, *An integral representation of a $_{10}\phi_9$ and continuous bi-orthogonal $_{10}\phi_9$ rational functions*, Can. J. Math. **38** (1986), 605–618.
- R. E. M. Rains, *Transformations of elliptic hypergeometric integrals*, preprint (2003).
- Ru1. S. N. M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997), 1069–1146.
- Ru2. ———, *Generalized hypergeometric function satisfying four analytic difference equations of Askey-Wilson type*, Commun. Math. Phys. **206** (1999), 639–690.
- Ru3. ———, *A generalized hypergeometric function III. Associated Hilbert space transform*, Commun. Math. Phys. **243** (2003), 413–448.
- Sh. T. Shintani, *On a Kronecker limit formula for real quadratic field*, J. Fac. Sci. Univ. Tokyo **24** (1977), 167–199.
- S1. V. P. Spiridonov, *An elliptic beta integral*, Proc. Fifth International Conference on Difference Equations and Applications (Temuco, Chile, January 3–7, 2000), Taylor and Francis, London, 2001, pp. 273–282; *On the elliptic beta function*, Russ. Math. Surveys **56** (1) (2001), 185–186.
- S2. ———, *Theta hypergeometric integrals*, Algebra i Analiz **15** (2003), 161–215. (St. Petersburg Math. J. **15** (2004), 929–967).
- S3. ———, *A Bailey tree for integrals*, Theor. Math. Phys. **139** (2004), 536–541.
- St. J. V. Stokman, *Hyperbolic beta integrals*, Adv. Math. **190** (2004), 119–160.
- T. Y. Takeyama, *The q -twisted cohomology and the q -hypergeometric function at $|q| = 1$* , Publ. Res. Inst. Math. Sci. **37** (2001), no. 1, 71–89.
- WW. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1986.

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 747, TALCA, CHILE

BOGOLIUBOV LABORATORY OF THEORETICAL PHYSICS, JOINT INSTITUTE FOR NUCLEAR RESEARCH, DUBNA, MOSCOW REGION 141980, RUSSIA